MAA
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AIME I

Solutions Pamphlet

American Invitational Mathematics Examination I Solutions Pamphlet
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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:

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The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

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1. (Answer: 790)

The lace must be long enough to pass along one width of the rectangle and six diagonal crisscrosses, and include the two loose ends for tying the knot. The width is 50 mm. The diagonals are hypotenuses of right triangles with legs measuring $\frac{80}{3}$ mm and 50 mm. Because $50 = \frac{150}{3}$, these numbers are proportional to the first two terms of the Pythagorean triple 8, 15, 17, and the diagonal length is $\frac{170}{3}$ mm. The total length required is therefore $50 + 6 \cdot \frac{170}{3} + 2 \cdot 200 = 790$ mm.

2. (Answer: 144)

The event "both balls have the same color" is the union of two disjoint events, "ball 1 and ball 2 are both green" and "ball 1 and ball 2 are both blue." Because the selections from the two urns are independent, the probability that the two balls are the same color is

$$\begin{aligned} &P(\text{ball 1 green}) \cdot P(\text{ball 2 green}) + P(\text{ball 1 blue}) \cdot P(\text{ball 2 blue}) \\ &\qquad = \frac{4}{10} \cdot \frac{16}{16+N} + \frac{6}{10} \cdot \frac{N}{16+N} \\ &\qquad = .58. \end{aligned}$$

Multiplying by $100(16+N)$ yields $40 \cdot 16 + 60 \cdot N = 58(16+N)$ which reduces to $N = 144$.

3. (Answer : 200)

Let $r = \frac{a}{b}$ be a rational number in lowest terms with $a + b = 1000$. If $\gcd(a, 1000) = d > 1$, then d is also a factor of $b = 1000 - a$ so $\frac{a}{b}$ is not in lowest terms. On the other hand if $\gcd(a, 1000) = 1$ and $b = 1000 - a$, then $\gcd(a, b) = 1$, and $\frac{a}{b}$ is in lowest terms. Because $1000 = 2^3 \cdot 5^3$, the number of positive integers less than 1000 and relatively prime to 1000 is

$$\phi(1000) = 1000 - \frac{1000}{2} - \frac{1000}{5} + \frac{1000}{10} = 400.$$

Half of these numbers (i.e. 200) are less than $\frac{1}{2} \cdot 1000$, and these are the possible numerators for a fraction of the desired type.

4. (Answer: 049)

Let the length of each train be L miles. In passing each rider, each train travels L miles relative to that rider. Because the trains each go past Jon in 1 minute, their speed relative to Jon is L miles per minute. Jon and Steve are each riding at a speed of $\frac{1}{3}$ mile per minute in opposite directions. Therefore, relative to Steve, the speed of the eastbound train is $L + \frac{2}{3}$ miles per minute, and the speed of the westbound train is $L - \frac{2}{3}$ miles per minute. The times required for the trains to go past Steve are $\frac{L}{L+\frac{2}{3}} = \frac{3L}{3L+2}$ and $\frac{L}{L-\frac{2}{3}} = \frac{3L}{3L-2}$, respectively. Thus $\frac{3L}{3L-2} = 10\left(\frac{3L}{3L+2}\right)$, from which $L = \frac{22}{27}$. The requested sum is $22 + 27 = 49$.

5. (Answer: 134)

It is clear that S and the empty set are both communal. Let C be the circle that passes through the points of S. A proper subset Q of S is communal if and only if there is an arc of C such that Q is the set of points of S that lie on that arc. To see this let Q be a communal subset of S, and let D be a circle so that the points of Q are inside of D, and the points of S not in Q are outside of D. Because the portion of C that is inside of D is an arc, the points of Q must be the points of S that lie on this arc.

Now let Q be the set of points in S that lie on an arc α of C. If necessary, extend α slightly so that no points of S are endpoints of α. Let A and B be the endpoints of α, and let M be the midpoint of α. Let D be the circle with center M passing through A and B. Then α is the arc of C inside of D, all points of Q are inside of D, and all points of S not in Q are outside of D. This shows that Q is communal.

Let $1 \le k \le 11$, and let Q be a communal subset with k points. Then each rotation of Q through an angle of $\frac{360}{12} = 30^\circ$ results in another communal subset of k elements. With successive rotations this process generates 12 communal subsets of k elements. Thus the number of communal subsets is $11 \cdot 12 + 2 = 134$.

6. (Answer: 036)

The first equation can be rewritten as $y = 3x^2 - 6xh + 2013$. If its x-intercepts are p and q, then $pq = \frac{2013}{3} = 671 = 11 \cdot 61$, so $\{p, q\} = \{1, 671\}$ or $\{11, 61\}$. The value of h is the average of p and q, so $h = 336$ or 36. The second equation can be rewritten as $y = 2x^2 - 4xh + 2014$. If its x-intercepts are r and s, then $rs = \frac{2014}{2} = 1007 = 19 \cdot 53$, so $\{r, s\} = \{1, 1007\}$ or $\{19, 53\}$, and $h = 504$ or 36. The only value of h that is possible for both equations is 36. Note that $y = 3(x - 36)^2 - 1875$ and $y = 2(x - 36)^2 - 578$ satisfy the conditions in the problem.

7. (Answer: 100)

Consider a geometric interpretation. Point $A = w$ is on the unit circle centered at the origin O. Point $B = z$ is on the circle with radius 10 centered at the origin. Because dividing two complex numbers subtracts their complex arguments, θ is the angle between z and the vector from w to z, that is, $\angle ABO$. The tangent of this angle is clearly maximized when $\angle ABO$ is maximized which occurs when $\overline{AB}$ is tangent to the unit circle at A. When this happens, $\triangle ABO$ is a right triangle with right angle $\angle OAB$. Then the Pythagorean Theorem shows $AB^2 =$

$OB^2 - OA^2 = 10^2 - 1^2 = 99$. It follows that $\tan^2\theta = \frac{OA^2}{AB^2} = \frac{1}{99}$. The requested sum is $1 + 99 = 100$.

8. (Answer: 937)

   Write $N = 10000M + k$, where M is an integer, and $1000 \le k \le 9999$. Then because N^2 and N end in the same four digits,

   $$N^2 - N = (10^4M + k)^2 - (10^4M + k) = 10^4(10^4M^2 + 2Mk - M) + (k^2 - k)$$

   ends in four zeroes. Thus $10000 = 2^4 \cdot 5^4$ divides $k^2 - k = k(k-1)$. Because k and $k-1$ are relatively prime, there are four possibilities.

   - $10^4 \mid k$ so $k = 0$,
   - $10^4 \mid (k-1)$ so $k = 1$,
   - $5^4 \mid k$ and $2^4 \mid (k-1)$ so $k = 625$, and
   - $2^4 \mid k$ and $5^4 \mid (k-1)$ so $k-1$ must be $625m$ for a positive integer m where 16 divides $625m + 1$. Because $625 \equiv 1 \pmod{16}$, $15 \cdot 625 \equiv 15 \pmod{16}$ and 16 divides $15 \cdot 625 + 1 = 9376$ implying that $k = 9376$.

   The first three cases are not possible if $k \ge 1000$, but $k = 9376$ does work. The requested three digit number is 937.

   Note that for any positive integer r, if the positive integers N and N^2 end in the same r digits, there are always exactly 4 possible sequences of r final digits.

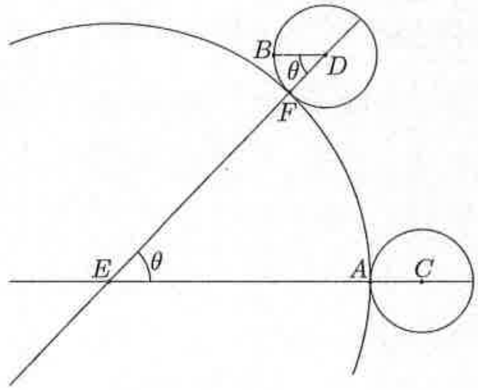
9. (Answer: 002)

   Let $y = \sqrt{2014}$. Then the equation becomes $0 = yx^3 - (2y^2+1)x^2 + 2 = -2x^2y^2 + x^3y - (x^2 - 2)$, a quadratic equation in y. The quadratic formula gives solutions $y = \frac{1}{x}$ or $y = \frac{x}{2} - \frac{1}{x}$. In the first case $x = \frac{1}{y} = \frac{1}{\sqrt{2014}}$. In the second case $x^2 - 2\sqrt{2014}x - 2 = 0$ which has solutions $x = \sqrt{2014} \pm \sqrt{2016}$. Thus $x_1 = \sqrt{2014} - \sqrt{2016}$, $x_2 = \frac{1}{\sqrt{2014}}$, and $x_3 = \sqrt{2014} + \sqrt{2016}$.

   Finally, the requested computation is $x_2(x_1 + x_3) = 2$.

10. (Answer: 058)

    Let F be the point on the circumference of the larger disk where the smaller disk is tangent after it has rolled through $360°$. Let $\theta = \angle FEA$. Because line $\overline{BD}$ is parallel to line $\overline{AC}$, $\angle FDB$ is also θ. The arc along the larger disk from A to F is the same length as the arc on the smaller disk whose central angle is the reflex $\angle FDB$. That central angle is therefore 5θ, and $360° = 6\theta$ implying $\theta = 60°$. The Law of Cosines applied to $\triangle BDE$ shows that $BE^2 = BD^2 + DE^2 - 2 \cdot BD \cdot DE \cdot \cos\theta = 1^2 + 6^2 - 2 \cdot 1 \cdot 6 \cdot \frac{1}{2} = 31$. The distance from B to line EA is $DE \cdot \sin\theta = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$. Thus $\sin^2(\angle BEA) = \frac{(3\sqrt{3})^2}{BE^2} = \frac{27}{31}$. The requested sum is $27 + 31 = 58$.



11. (Answer: 391)

By symmetry, the same number of sequences of moves will end in each of the four quadrants of the coordinate plane. Let r, l, u, and d represent the number of moves made in the positive x-direction, the negative x-direction, the positive y-direction, and the negative y-direction, respectively.

The token ends at $(3,3)$ if and only if $(r,l,u,d)=(3,0,3,0)$. The number of such sequences of moves is $\frac{6!}{3!\cdot0!\cdot3!\cdot0!}=20$.

The token ends at $(2,2)$ if and only if $(r,l,u,d)=(2,0,3,1)$ or $(3,1,2,0)$. The number of such sequences of moves is $2\cdot\frac{6!}{2!\cdot0!\cdot3!\cdot1!}=120$.

The token ends at $(1,1)$ if and only if $(r,l,u,d)=(1,0,3,2),(2,1,2,1)$, or $(3,2,1,0)$. The number of such sequences of moves is $2\cdot\frac{6!}{1!\cdot0!\cdot3!\cdot2!}+\frac{6!}{2!\cdot1!\cdot2!\cdot1!}=300$.

The token ends at $(0,0)$ if and only if $(r,l,u,d)=(0,0,3,3),(1,1,2,2),(2,2,1,1)$, or $(3,3,0,0)$. The number of such sequences of moves is $2\left(\frac{6!}{0!\cdot0!\cdot3!\cdot3!}+\frac{6!}{1!\cdot1!\cdot2!\cdot2!}\right)=400$.

Thus the number of sequence of moves that end on the graph of $|y|=|x|$ is $400+4(20+120+300)=2160$. The total number of six-move sequences possible is 4^6, so the sequence ends on the graph of $|y|=|x|$ with probability $\frac{2160}{4^6}=\frac{135}{4^4}=\frac{135}{256}$. The requested sum is $135+256=391$.

12. (Answer: 453)

For each of the 4 one-element subsets $B\subseteq A$ there is $1^4=1$ function on A whose range is B, so there are 4 functions on A with one-element ranges. For each of the $\binom{4}{2}=6$ two-element subsets $B\subseteq A$, there are $2^4-2=14$ functions on A whose range is B, so there are $6 \cdot 14 = 84$ functions on A with two-element ranges. For each of the $\binom{4}{3} = 4$ three-element subsets $B \subseteq A$, there are $3^4 - 3 \cdot 14 - 3 = 36$ functions on A whose range is B, so there are $4 \cdot 36 = 144$ functions on A with three-element ranges. It follows that the number of ordered pairs (f, g) of functions on A whose ranges are disjoint is $4 \cdot 3^4 + 84 \cdot 2^4 + 144 \cdot 1^4 = 4\,(81 + 336 + 36) = 4 \cdot 453$. The probability that such a pair of functions is chosen at random is $\frac{4 \cdot 453}{4^4 \cdot 4^4} = \frac{453}{2^{14}}$. The requested numerator is 453.

13. (Answer: 850)

Let s be the side length of $ABCD$, let Q and R be the midpoints of $\overline{EG}$ and $\overline{FH}$, respectively, let S be the foot of the perpendicular from Q to $\overline{CD}$, and let T be the foot of the perpendicular from R to $\overline{AD}$. The fraction of the area of the square $ABCD$ which is occupied by trapezoid $BCGE$ is

$$\frac{275 + 405}{269 + 275 + 405 + 411} = \frac{1}{2},$$

so Q is the center of $ABCD$. Thus R, Q, and S are collinear, and $RT = QS = \frac{1}{2}s$. Similarly, the fraction of the area occupied by trapezoid $CDHF$ is $\frac{3}{5}$, so $RS = \frac{3}{5}s$ and $RQ = \frac{1}{10}s$.

Because $\triangle QSG \cong \triangle RTH$, the area of $DHPG$ is the sum of the areas of rectangle $DTRS$ and $\triangle RPQ$. Rectangle $DTRS$ has area $RS \cdot RT = \frac{3}{5}s \cdot \frac{1}{2}s = \frac{3}{10}s^2$. If $\theta = \angle QRP$, then $\triangle RPQ$ has area $\frac{1}{2} \cdot \frac{1}{10}s \sin\theta \cdot \frac{1}{10}s\cos\theta = \frac{1}{400}s^2 \sin 2\theta$. Therefore the area of $DHPG$ is $s^2(\frac{3}{10} + \frac{1}{400}\sin 2\theta)$. Because the area of trapezoid $CDHF$ is $\frac{3}{5}s^2$, the area of $CGPF$ is $s^2(\frac{3}{10} - \frac{1}{400}\sin 2\theta)$. Because these areas are in the ratio $411 : 405 = (408 + 3) : (408 - 3)$, it follows that

$$\frac{\frac{1}{400}\sin 2\theta}{\frac{3}{10}} = \frac{3}{408},$$

from which $\sin 2\theta = \frac{15}{17}$. Note that $\theta = \angle RHT > \angle QAT = 45°$, so $\cos 2\theta = -\sqrt{1 - \sin^2 2\theta} = -\frac{8}{17}$ and $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta) = \frac{25}{34}$. The area of $ABCD$ is $s^2 = EG^2 \sin^2\theta = 34^2 \cdot \frac{25}{34} = 850$.

14. (Answer: 263)

Because $\frac{a}{x-a} = \frac{x}{x-a} - 1$, the given equation is equivalent to

$$x(x-11) = \frac{x}{x-3} + \frac{x}{x-17} + \frac{x}{x-5} + \frac{x}{x-19}.$$

Thus $x = 0$ is a solution of the equation. If $y = x - 11$, then the rest of the solutions satisfy

$$y = \frac{1}{y+8} + \frac{1}{y-8} + \frac{1}{y+6} + \frac{1}{y-6} = \frac{2y}{y^2-64} + \frac{2y}{y^2-36}.$$

Thus $y = 0$ (that is, $x = 11$) is a solution of the equation. If $z = y^2 - 50$, then the rest of the solutions satisfy

$$1 = \frac{2}{z - 14} + \frac{2}{z + 14},$$

from which it follows that $z^2 - 14^2 = 4z$. Therefore $z = \pm\sqrt{200} + 2$, and

$$(x - 11)^2 = y^2 = z + 50 = 52 \pm \sqrt{200},$$

and

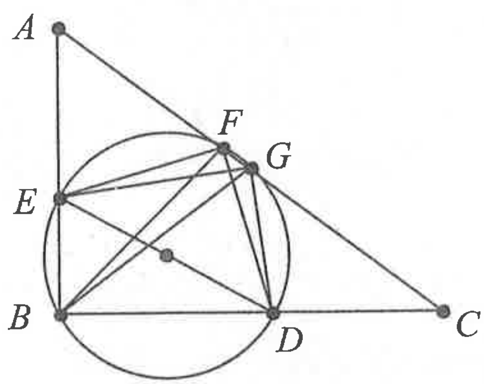
$$x = 11 \pm \sqrt{52 \pm \sqrt{200}},$$

implying that $m = 11 + \sqrt{52 + \sqrt{200}}$. The requested sum is $11 + 52 + 200 = 263$.

15. (Answer: 041)

Because points B, D, E, F, G all lie on ω and $\angle DBE = 90^\circ$, $\overline{DE}$ is a diameter of ω. It follows that $\angle DFE = \angle DGE = 90^\circ$.

Because $DF = EF$, $\triangle DFE$ is an isosceles right triangle and arc DF = arc EF, implying that $\angle DBF = \angle EBF = 45^\circ$; that is, $\overline{BF}$ bisects $\angle ABC$. In particular, by the Angle Bisector Theorem, $AF < FC$.



Because $\frac{GD}{EG} = \frac{AB}{BC}$ and $\angle DGE = \angle ABC = 90^\circ$, conclude that $\triangle DGE$ and $\triangle ABC$ are similar. In particular, $\angle GED = \angle BCA$ and $\angle GDE = \angle BAC$. Note also that $\angle GED = \angle GBD$ and $\angle GDE = \angle GBE$ because $BDGE$ is cyclic. Hence $\angle GBC = \angle GBD = \angle GED = \angle BCA = \angle BCG$ and $\angle GBA = \angle GBE = \angle GDE = \angle BAC = \angle BAG$, from which it follows that both $\triangle ABG$ and $\triangle CBG$ are isosceles with $AG = BG = CG$, implying that G is the midpoint of side AC.

Note that $AF < FC$ and $AG = GC$. Hence $BDGFE$ is a convex cyclic pentagon. Because $DEFG$ is cyclic, $\angle DGC = \angle DEF = 45^\circ$. The Extended Law of Sines and addition formulas imply

$$DE = \frac{BG}{\sin(\angle BDG)} = \frac{BG}{\sin(\angle CGD + \angle DCG)}$$

$$= \frac{\frac{5}{2}}{\frac{\sqrt{2}}{2}\left[\sin(\angle DCG) + \cos(\angle DCG)\right]} = \frac{5\sqrt{2}}{2\left(\frac{3}{5} + \frac{4}{5}\right)} = \frac{25\sqrt{2}}{14},$$

and $a = 25$, $b = 2$, and $c = 14$. The requested sum is $25 + 2 + 14 = 41$.

The problems and solutions in this contest were proposed by Alex Anderson, Steve Blasberg, Steve Dunbar, Zuming Feng, Peter Gilchrist, Elgin Johnston, Jonathan Kane, Cap Khoury, Tamas Szabo, Kevin Wang, Dave Wells, Ronald Yannone.